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On mimicry among sequential sampling models

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HIGHLIGHTS

- We prove the mimicry of a Wiener process by an independent race model.
- We examine the numerical computation of the mimicking boundaries.
- We show that the mimicking boundaries are time-varying and asymmetric.
- We propose an equivalent symmetric race model.
- We examine the mimicry of full diffusion model.

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ABSTRACT

Sequential sampling models are widely used in modeling the empirical data obtained from different decision making experiments. Since 1960s, several instantiations of these models have been proposed. A common assumption among these models is that the subject accumulates noisy information during the time course of a decision. The decision is made when the accumulated information favoring one of the responses reaches a decision boundary. Different models, however, make different assumptions about the information accumulation process and the implementation of the decision boundaries. Comparison among these models has proven to be challenging. In this paper we investigate the relationship between several of these models using a theoretical framework called the inverse first passage time problem. This framework has been used in the literature of applied probability theory in investigating the range of the first passage time distributions that can be produced by a stochastic process. In this paper, we use this framework to prove that any Wiener process model with two time-constant boundaries can be mimicked by an independent race model with time-varying boundaries. We also examine the numerical computation of the mimicking boundaries. We show that the mimicking boundaries of the race model are not symmetric. We then propose an equivalent race model in which the boundaries are symmetric and time-constant but the drift coefficients are time-varying.

1. Introduction

In the last few decades, a large amount of research has investigated the mechanisms underlying simple perceptual decision making. The basic idea is to examine how the subjects' reaction time and accuracy change as a function of the properties of noisy stimuli. Describing the pattern of this empirical data computationally has proven to be a challenging task. A "good" computational model should be able to describe the relation between the physical properties of the stimulus (e.g., the salience and discriminability) and the shape of the reaction time distributions, the accuracy, the relative speed of the correct and incorrect responses and the effect of emphasizing speed or accuracy in the instructions. Neurophysiological data obtained from the activity of populations of neurons during perceptual decision making experiments impose more restrictions on the computational models. One class of models which has been successful in accounting for these patterns of data is *sequential sampling models*. In this modeling framework, it is assumed that after the presentation of the stimulus, the subject starts accumulating noisy information favoring each alternative response in the task. The subject responds in a trial when the accumulated information favoring one of the alternatives reaches a specific amount called the *decision threshold*.

Several instantiations of this framework have been proposed by researchers including the full diffusion model (Ratcliff, 1978), Ornstein–Uhlenbeck (OU) model (Busemeyer & Townsend, 1993), leaky competing accumulator (LCA) model (Usher & McClelland,

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2001), linear ballistic accumulator (LBA) model (Brown & Heathcote, 2008), race models (Eidels, Houpt, Altieri, Pei, & Townsend, 2011; Smith & Vickers, 1988; Townsend & Ashby, 1983) and accrual halting models (Townsend, Houpt, & Silbert, 2012). These models differ in their assumptions about the information accumulation process and the way that the decision is made based on this information.

Comparison among these models poses another challenge to the computational modeling of perceptual decision making. Model comparison is particularly challenging because in many situations these models make similar predictions. Two general approaches have been employed by researchers to compare these models. In the first approach, the models are fitted to the empirical data and are compared based on some statistical measures of *goodness of fit*, for example, chi-square, sum of squared errors, BIC and AIC (Ratcliff & Smith, 2004; Ratcliff & Tuerlinckx, 2002; Ratcliff, Van Zandt, & McKoon, 1999; Tsetsos, Gao, McClelland, & Usher, 2012; Van Zandt, Colonius, & Proctor, 2000). Besides quantitative fit, the qualitative predictions of each model are compared to the patterns in the data. For example, a common finding in reaction time experiments is that the mean reaction times for the correct and incorrect responses are not the same. Any model that cannot predict this pattern is not likely to be a good model of these experimental data.

The second approach is to examine the theoretical relationship between these models without considering the data (Dzhafarov, 1993; Jones & Dzhafarov, 2014; Pike, 1968; Smith, 2010; Townsend, 1976; Townsend & Ashby, 1983; Zhang, Lee, Vandekerckhove, Maris, & Wagenmakers, 2014). Of particular interest in this vein of research is the problem of *model mimicry*. Two models of the reaction time mimic each other if they produce the exact same distributions of reaction time. The research in this area is less prevalent. One main reason is that, besides a few exceptions, the analytic form of the distributions of reaction time predicted by these models is not known. Therefore, it is hard to determine the range of patterns that can be produced by each model. This is especially the case when the accumulation process is modeled as a stochastic process. Recently, Jones and Dzhafarov (2014) have theoretically investigated the range of reaction time distributions that can be produced by several classes of models. In the models considered in their paper, neither the accumulation process nor the decision thresholds are stochastic processes. Instead, the models consist of random variables and deterministic time-varying functions.[1] More recently, Zhang et al. (2014) proposed a new method for investigating the mimicry between sequential sampling models in which the accumulation process is a stochastic process and the decision thresholds are time-varying functions. In their method, the problem of mimicking a model by another model is translated into another problem called *the inverse first passage time problem*. A model can mimic another one if the corresponding inverse first passage time problem is solvable. The authors considered the mimicry between a diffusion model and an accumulator model. They showed how one can numerically compute two time-varying boundaries for a diffusion model such that it mimics an accumulator model with symmetric boundaries. Although their simulation results suggest that an accumulator model can always be mimicked by a diffusion model, no theoretical analysis is provided in the paper.

In this paper, we take the same approach for investigating mimicry among sequential sampling models. Specifically, we consider the following question: can a Wiener process with constant boundaries be mimicked by an independent race model? The main goal of this paper is to investigate this question theoretically using the existing theorems in the stochastic processes literature, particularly the inverse first passage time problems. To this end, in the following two sections we introduce the stochastic processes considered in this paper and give a formal definition of the inverse first passage time problem. Then, in Section 4 we review some of the existing theorems regarding the inverse first passage time problem that we will use in deriving our results. In Section 5, we present our theoretical results on mimicry between the Wiener process model and the independent race model and then in Section 6 the numerical results are reported. In Section 7, we compare our results with some of the theoretical results in Jones and Dzhafarov (2014). Finally, Section 9 is devoted to the problem of mimicry of a Wiener process model by an OU process model.

2. Wiener process and independent race models of decision making

As explained in the Introduction, in a sequential sampling model it is assumed that the information favoring each alternative is accumulated and the subject responds in a trial whenever the accumulated information reaches a decision boundary. In a Wiener process model (also known as the Wiener diffusion model) of a task with two alternatives, the accumulated information is modeled as a stochastic process called the Wiener process. Formally, a Wiener process $X(t)$ is characterized by the following stochastic differential equation (SDE):

$$dX(t) = \mu \cdot dt + \sigma \cdot dB(t). \quad (1)$$

In this equation, the parameters μ and σ are called the drift and the diffusion coefficients, respectively. It can be shown that $\mathrm{E}[\mathrm{X(t)}] = \mu \cdot t$ and $\mathrm{Var}[\mathrm{X(t)}] = \sigma^2 \cdot t$ and so these parameters determine the mean and the variance of the process at each time (see for example Smith, 2000). The process dB specifies the increments of a zero-mean Gaussian process. In this paper, we always assume that the initial value of the process is zero ($X(0) = 0$).

In this model, it is assumed that the response 1 (response 2) is chosen in a trial if the process exceeds the decision boundary b_1 (b_2) before it hits the other decision boundary b_2 (b_1). In the literature of the stochastic processes, the first time that the process hits a decision boundary is called the first passage time (FPT). In the sequential sampling models, the FPT of the process is considered as the subject's decision time. Formally, the FPTs for the decision boundaries b_1 and b_2 are defined as follows:

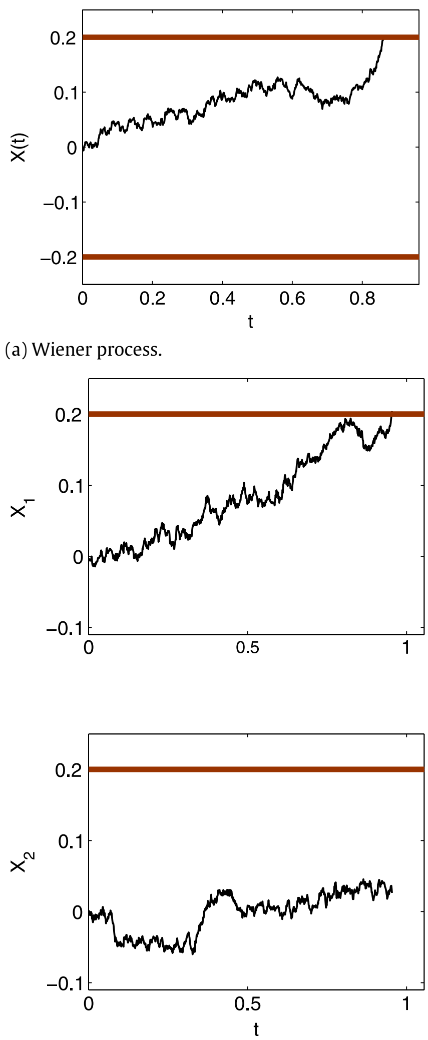
$$\begin{aligned} T_1 &= \inf\{t > 0 | X(t) \geq b_1 \text{ AND } X(\tau) > b_2, \text{ for all } \tau < t\} \\ T_2 &= \inf\{t > 0 | X(t) \leq b_2 \text{ AND } X(\tau) < b_1, \text{ for all } \tau < t\}. \end{aligned} \quad (2)$$

Because of the noise term dB in Eq. (1), the FPTs T_i are random variables. The subject's reaction time in each trial is a realization of either of these two random variables. We assume that when the accumulated information of one accumulator reaches its threshold first in a trial, the FPT in the other accumulator is ∞. To this end we adopt the convention that the infimum of an empty set is infinity.

We denote the probability density function (p.d.f.) of T_i by $g_i(t)$ (that is, $g_i(t) = \frac{d}{dt}\Pr(T_i \leq t)$). It is important to note that $\int_0^\infty g_i(t)dt = P_i$, where P_i is the probability of choosing the response i. This probability is not necessarily 1 and so $g_i(t)$ is a *defective* probability density function. A sample path of a Wiener process along with two decision boundaries is shown in the upper panel of Fig. 1.

Another sequential sampling model that we consider in this paper is the *independent race model*. In an independent race model there is a separate information accumulator for each alternative response. Each accumulator is modeled as a stochastic process. These

[1] Even in the Wiener processes considered in Theorems 11 and 12 in Jones and Dzhafarov (2014) the signal to noise ratio should be so large that the information accumulation reduces to a deterministic growth rate (see Smith, Ratcliff, & McKoon, 2014).



(a) Wiener process.

(b) Independent race model.

Fig. 1. (a) A sample path of a Wiener process. (b) A sample path of an independent race model. The thin black lines indicate the amount of accumulated information and the thick lines indicate the decision thresholds. In panel (a), since the accumulated information has reached b_1 before b_2, response 1 will be chosen and the decision time is about 0.85 s. In panel (b), since X_1 has reached its boundary sooner, the response corresponding to the first accumulator will be chosen and the decision time is about 0.95 s.

processes are assumed to be independent of each other and each of them has its own decision boundary. In each trial, the response corresponding to the process that first reaches its boundary is chosen and the decision time is equal to the FPT of that process. The lower panel of Fig. 1 shows a sample path of an independent race model. In this figure, it is assumed that there are two possible responses and so there are two processes X_1 and X_2 corresponding to them. Since the process X_1 has reached its boundary sooner, its corresponding response is chosen.

The decision boundaries in the models in Fig. 1 are constant. However, we can extend the definition of the FPT to the case of time-varying boundaries. Let $b_i(t)$, $i = 1, 2$ denote the time-varying decision boundary for the ith accumulator. The FPT of the process X_i through the boundary $b_i(t)$ (ignoring the other accumulator) is defined as follows:

$$T_i = \inf\{t > 0 | X_i(t) \geq b_i(t)\}. \tag{3}$$

We denote the p.d.f. of T_i by $f_i(t)$. It is important to note that, for example, T_1 is the FPT of X_1 through b_1 without considering the process X_2 (T_2 is defined similarly). However, in an independent race model we are interested in the first time that, for example, the process X_1 reaches its boundary *before* the process X_2 reaches its boundary. Let $\widehat{T_i}$, $i = 1, 2$ denote the time at which the process i reaches its boundary before the other process does so. Since the process that first reaches its boundary determines the response and the decision time, $\widehat{T_i}$ is the decision time for response i. Formally, we can define the random variables $\widehat{T_i}$, $i = 1, 2$ as follows:

$$\begin{aligned} \widehat{T_1} &= \inf\{t > 0 | X_1(t) \geq b_1(t) \text{ AND } X_2(\tau) \\ &< b_2(\tau) \text{ for all } \tau < t\} \\ \widehat{T_2} &= \inf\{t > 0 | X_2(t) \geq b_2(t) \text{ AND } X_1(\tau) \\ &< b_1(\tau) \text{ for all } \tau < t\}. \end{aligned} \tag{4}$$

Let $h_i(t)$ denote the p.d.f. of $\widehat{T_i}$. It is easy to show that:

$$\begin{aligned} h_1(t) &= f_1(t) \cdot S_2(t) \\ h_2(t) &= f_2(t) \cdot S_1(t) \end{aligned} \tag{5}$$

where $S_i(t) = 1 - \int_0^t f_i(\tau)d\tau = \int_t^\infty f_i(\tau)d\tau$ is the *survivor function* of the random variable T_i. It is worth noting that $\int_0^\infty f_i(t)dt = 1$[2] but $\int_0^\infty h_i(t)dt = P_i$ where P_i is the probability of choosing response i, which could be less than 1 and so $h_i(t)$ is a defective p.d.f.

The processes X_1 and X_2 in an independent race model can be any stochastic process. However, from now on we confine ourselves to the case where they are Wiener processes and so the model is specified with the parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2)$ and the two boundaries b_1 and b_2. Throughout this paper, we use the name independent race model to refer to this restricted version of the independent race models.[3]

3. Direct and inverse first passage time problems

Following Zhang et al. (2014), to investigate the relationship between the Wiener process model and the independent race models we will consider the corresponding inverse FPT problem. In this section, we give a formal definition of the direct and the inverse FPT problems.

Consider a Wiener process X with parameters (μ, σ). Now consider a function $b : (0, \infty) \to \mathbb{R}$ satisfying $b(0+) > 0$ (that is, the limit of $b(t)$ as t approaches zero from above is positive). The FPT of the process X through the boundary $b(t)$ is $T = \inf\{t > 0 | X_t \geq b(t)\}$. Let $f(t)$ denote the p.d.f. of the FPT, that is $f(t) = \frac{d}{dt}\Pr(T \leq t)$. Regarding the relationship between the process X, the boundary $b(t)$ and the p.d.f. $f(t)$, two problems can be considered: the direct FPT problem and the inverse FPT problem.

Direct FPT problem. Given a process X with specific parameters (μ, σ) and the boundary $b(t)$ for all time t, the problem is to determine the p.d.f. $f(t)$. In other words, here we seek the distribution of the FPT of the process through the decision boundary. This problem is familiar to mathematical psychologists. Several numerical methods have been proposed for computing the p.d.f.s of the FPTs of a given Wiener process with one or two boundaries (Brown, Ratcliff, & Smith, 2006; Diederich & Busemeyer, 2003; Smith, 2000; Voss & Voss, 2008). When the boundaries are constant (i.e., $b(t) = b = constant$) the p.d.f.s can be expressed as analytic functions of the

[2] It is important to note that $\int_0^\infty f_i(t)dt$ does not always equal 1. For example, for a constant boundary b, this condition is true only if μ and b have the same sign (see Karatzas & Shreve, 1991, Chapter 3, for a proof of this). However, as we will see (Eq. (14)), in this paper we will only encounter situations in which this condition is true.

[3] Ratcliff, Hasegawa, Hasegawa, Smith, and Segraves (2007) called an independent race model with Wiener process accumulators, a dual diffusion model.

parameters (μ, σ, b) and time (see for example Ratcliff & Smith, 2004 Eqs. (A2a) and (A2b)).

Inverse FPT problem. Given the process X with specific parameters (μ, σ) and a given p.d.f. $g_d(t)$, determine the decision boundary $b(t)$ such that $f(t) = g_d(t)$. In other words, in this problem for a given process we must compute a boundary $b(t)$ such that the FPT of the process through that boundary is equal to a desired density $g_d(t)$. Since the density is given, the problem is called inverse (Capocelli & Ricciardi, 1972; Cheng, Chen, Chadam, & Saunders, 2006; Zucca & Sacerdote, 2009). For a process with two decision boundaries this problem is defined similarly.

There is little theoretical work on the inverse FPT problem. Even in the literature of applied probability and statistics, most of the papers published on this topic have focused on the numerical computation of the boundary. One important theoretical question in the inverse FPT problem is that for a given process, for what forms of the desired p.d.f. $g_d(t)$, does there exist a decision boundary $b(t)$ such that $f(t) = g_d(t)$? It is also important to know if this boundary is unique. In the next section, we summarize some of the previous theoretical results concerning these questions.

4. Related theoretical results

Capocelli and Ricciardi (1972) were among the first researchers who studied the inverse FPT problem theoretically. They investigated the conditions under which a given function can be considered as the p.d.f. of the FPT of a continuous, time-homogeneous and one-dimensional Markov process through a constant boundary.[4] Specifically, they showed that for a given family of functions $f(b, t|x_0)$, there exists *at most one* continuous, 1-dimensional and time-homogeneous Markov process such that $f(b, t|x_0)$ is the p.d.f. of the FPT of that process through the boundary b for all values of the parameter b in an interval of real numbers.

To the best of our knowledge, Cheng et al. (2006) were the first to consider the question of the existence and uniqueness of the solution of an inverse FPT problem. The proof of our theorem in the next section is based on their results and so here we restate the main theorem in their paper.

Theorem (*Cheng et al., 2006*)**.** *Consider a diffusion process defined by the stochastic differential equation:*

$$dX(t) = \Psi(X(t), t) \cdot dt + \Phi(X(t), t) \cdot dB(t) \tag{6}$$

where smooth and bounded functions $\Psi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}$ *and* $\Phi : \mathbb{R} \times \mathbb{R}_+ \to \mathbb{R}+$ $(\Phi(X, t) > \epsilon > 0)$ *are given. Also, consider a given cumulative distribution function (CDF)* $F(t) = 1 - S(t)$ *with:*

$$\lim_{t\to 0} S(t) = S(0) = 1, \qquad S(t_1) \geq S(t_2) \geq 0 \quad \forall t_1 < t_2. \tag{7}$$

Then, there exists *a* unique *function* $b(t)$ *such that* $F(t)$ *is the CDF of the FPT of the process* $X(t)$ *through the boundary* $b(t)$.

In other words, for any process specified by Eq. (6) with the given parameters, and any given well-defined CDF $F(t)$, the corresponding inverse FPT problem is solvable and the solution is unique.

Remark 1. For a given diffusion process and a given proper CDF, this theory shows the existence and the uniqueness of a *single* boundary that solves the inverse FPT problem. It is still an open research question that whether the same results are true for diffusion processes with two boundaries.

[4] Such processes are specified by the SDE $dX(t) = \Psi(X)dt + \Phi(X)dB(t)$, where $\Psi : \mathbb{R} \to \mathbb{R}$ and $\Phi : \mathbb{R} \to \mathbb{R}$ are smooth bounded functions.

Remark 2. The Wiener process is a special case of Eq. (6) with $\Psi(X, t) = \mu$ and $\Phi(X, t) = \sigma$.

Remark 3. The solution $b(t)$ is not necessarily continuous.

The importance of this theorem is readily obvious: it shows that by choosing an appropriate decision boundary, the p.d.f. of the FPT of a diffusion process can take any arbitrary form.

5. Mimicry of Wiener process model with symmetrical boundaries by independent race model

In this section, we first explain how the problem of mimicry between two models can be translated into an inverse FPT problem. This method was first proposed by Zhang et al. (2014). We then prove a theorem regarding the mimicry between the Wiener process model and the independent race model.

Consider a Wiener process model $M_1(\mu, \sigma, a)$ with given parameters (μ, σ) and two given decision boundaries $\pm a$. Also suppose that $X(0) = 0$. In this model the boundaries are symmetrical and the initial value of the accumulated information is equidistant from the two decision boundaries. Let $g_1(t)$ and $g_2(t)$ denote the p.d.f.s of the FPT of this process through the boundaries a and $-a$, respectively. We are interested in the following question: can an independent race model mimic this model? More precisely, can we find an independent race model such that $h_1(t) = g_1(t)$ and $h_2(t) = g_2(t)$ where $h_i(t)$ are defined in Eq. (5)? This question can be cast as an inverse FPT problem as follows: consider an independent race model with given parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2)$. The inverse FPT problem is to investigate the existence of two decision boundaries $b_1(t)$ and $b_2(t)$ such that the FPTs of this model through these boundaries are $h_1(t) = g_1(t)$ and $h_2(t) = g_2(t)$. The theorem below shows that for all values of (μ, σ, a) this inverse FPT is solvable and so the Wiener process model with symmetrical boundaries can be mimicked by the independent race model.

Theorem. *Consider a Wiener process with parameters* (μ, σ)*. Let* $g_1(t)$ *and* $g_2(t)$ *denote the p.d.f. of the FPT of this process through the boundaries* a *and* $-a$*, respectively, and let* $X(0) = 0$*. In addition, consider an independent race model with parameters* $(\mu_1, \mu_2, \sigma_1, \sigma_2)$*. Then,* (a) *for all values of* μ, μ_1, μ_2 *and all* $\sigma, \sigma_1, \sigma_2 \geq 0$*, there* exist *two* unique *functions* $b_1(t)$ *and* $b_2(t)$ *such that* $h_1(t) = g_1(t)$ *and* $h_2(t) = g_2(t)$. (b) *for all values of* $\mu, \sigma, \mu_1, \mu_2$ *and* a*, if* $\sigma_1 = \sigma_2 = \sigma$ *then we have* $\lim_{t\to 0} b_i(t) = a$.

In other words, for a given Wiener process model with arbitrary parameters and constant symmetrical boundaries, and a given independent race model with arbitrary values of the parameters, it is always possible to choose the boundaries of the independent race model in such a way that it mimics the Wiener process model.

Proof. We first take an approach similar to Townsend (1976) to derive the relationship between the p.d.f.s of FPT of the Wiener process model and the mimicking independent race model. For the independent race model to mimic the Wiener process model, the following equations relating the p.d.f.s of the FPT must hold:

$$\begin{aligned} g_1(t) = h_1(t) &\Longrightarrow f_1(t) = \frac{g_1(t)}{S_2(t)}, \quad \forall t \\ g_2(t) = h_2(t) &\Longrightarrow f_2(t) = \frac{g_2(t)}{S_1(t)}, \quad \forall t. \end{aligned} \tag{8}$$

If we can prove that the functions $f_i(t)$, $i = 1, 2$ obtained from these equations are well-defined, then Cheng et al.'s theorem, explained in the previous section, assures that there always exist unique functions $b_1(t)$ and $b_2(t)$ such that the p.d.f. of the FPT of the ith accumulator is $f_i(t)$.

In what follows, we prove that there always exist unique well-defined p.d.f.s $f_i(t)$ with corresponding survivor functions $S_i(t)$ that satisfy Eq. (8). Let $v_1(t) = \frac{g_1(t)}{p}$ and $v_2(t) = \frac{g_2(t)}{1-p}$ with $p = \int_0^\infty g_1(\tau)d\tau$ being the probability of choosing response i (note that $\int_0^\infty g_1(\tau)d\tau = 1 - \int_0^\infty g_2(\tau)d\tau$ and $\int_0^\infty v_i(\tau)d\tau = 1$). We have $p \cdot v_1(t) + (1-p) \cdot v_2(t) = g_1(t) + g_2(t) = f_1(t) \cdot S_2(t) + f_2(t) \cdot S_1(t)$. Taking the integral from t to ∞ of both sides of this equation we have:

$$\begin{aligned}\int_t^\infty & [p \cdot v_1(\tau) + (1-p) \cdot v_2(\tau)]d\tau \\ &= \int_t^\infty [f_1(\tau) \cdot S_2(\tau) + f_2(\tau) \cdot S_1(\tau)]d\tau \\ &= \int_t^\infty \left[-\frac{d}{d\tau}\Big(S_1(\tau) \cdot S_2(\tau)\Big)\right]d\tau \\ &= -S_1(\infty)S_2(\infty) + S_1(t)S_2(t). \end{aligned} \tag{9}$$

Now, assume that $S_1(\infty) \cdot S_2(\infty) = 0$ or equivalently $\lim_{t\to\infty} [S_i(t)] = 0$ for at least one of the accumulators.[5] This results in the following equation:

$$p \cdot \overline{V}_1(t) + (1-p) \cdot \overline{V}_2(t) = S_1(t) \cdot S_2(t) \tag{10}$$

where $\overline{V}_i(t) = \int_t^\infty v_i(\tau)d\tau$. Now we divide both sides of Eq. (8) by the two sides of Eq. (10) which yields:

$$\begin{aligned}\frac{f_1(t)}{S_1(t)} &= \frac{g_1(t)}{p \cdot \overline{V}_1(t) + (1-p) \cdot \overline{V}_2(t)} \\ \frac{f_2(t)}{S_2(t)} &= \frac{g_2(t)}{p \cdot \overline{V}_1(t) + (1-p) \cdot \overline{V}_2(t)}.\end{aligned} \tag{11}$$

Taking the integral from 0 to t of both sides of these equations gives:

$$\begin{aligned}-\ln[S_1(t)] &= \int_0^t \frac{g_1(\tau)}{p \cdot \overline{V}_1(\tau) + (1-p) \cdot \overline{V}_2(\tau)}d\tau \\ -\ln[S_2(t)] &= \int_0^t \frac{g_2(\tau)}{p \cdot \overline{V}_1(\tau) + (1-p) \cdot \overline{V}_2(\tau)}d\tau.\end{aligned} \tag{12}$$

The left hand side of these equations are obtained as follows: $\int_0^t \frac{f_i(\tau)}{S_i(\tau)d\tau} = -\ln[S_i(t)] + \ln[S_i(0)]$. Another necessary condition for well-defined $S_i(t)$ is that $S_i(0) = 1$ which results in the left hand side of Eq. (12).

Now we note that for a Wiener process if $X(0)$ is equidistant from its two decision boundaries then $\frac{g_1(t)}{g_2(t)} = \frac{p}{1-p}$ for all t. In this case $v_1(t) = v_2(t) = v(t)$. Substituting this in the right hand side of Eqs. (12) yields:

$$\begin{aligned}-\ln[S_1(t)] &= \int_0^t \frac{g_1(\tau)}{\overline{V}(\tau)}d\tau = \int_0^t \frac{p \cdot v(\tau)}{\overline{V}(\tau)}d\tau \\ &= -p \cdot \ln[\overline{V}(t)] \\ -\ln[S_2(t)] &= \int_0^t \frac{g_2(\tau)}{\overline{V}(\tau)}d\tau = \int_0^t \frac{(1-p) \cdot v(\tau)}{\overline{V}(\tau)}d\tau \\ &= -(1-p) \cdot \ln[\overline{V}(t)]\end{aligned} \tag{13}$$

[5] This condition implies that in the race model, in all trials at least one of the accumulators reaches its decision threshold and so the FPT in all trials is finite. In the proof, we show that even with this restriction the race model can mimic the Wiener process model with symmetrical boundaries.

and so:

$$\begin{aligned}S_1(t) &= [\overline{V}(t)]^p \\ S_2(t) &= [\overline{V}(t)]^{(1-p)}.\end{aligned} \tag{14}$$

It is easy to see that $\lim_{t\to\infty} S_i(t) = 0$, $S_i(0) = 1$ and $S_i(t)$ are differentiable and decreasing and so they are well-defined survivor functions. Therefore, if there exist the decision boundaries $b_i(t)$ such that the survivor functions of the random variables T_i (defined in Eq. (3)) are equal to those given by Eq. (14), then the given independent race model will mimic the given Wiener process model. Since the survivor functions given in Eq. (14) are well-defined, Cheng et al.'s theorem assures that for any given values of the parameters $(\mu, \sigma, a, \mu_1, \mu_2, \sigma_1, \sigma_2)$ such boundaries exist and they are unique and this completes the proof.

The proof of part (b) is presented in the Appendix. □

Remark 1. This theorem shows that the independent race model can mimic a Wiener process model with symmetrical boundaries. However, it is very important to note that the mimicry power of the independent race model is much more than this. Specifically, any pair $g_1(t)$ and $g_2(t)$ that result in well-defined survivor functions $S_i(t)$ in Eq. (12), can be mimicked by an independent race model. In the proof, we showed that this is the case when $g_1(t)$ and $g_2(t)$ are the p.d.fs of FPT of a Wiener process.

Remark 2. In the Theorem above we assumed that the accumulators of the independent race model are Wiener processes. However, since Cheng et al.'s theorem is true for all continuous Markov processes specified by Eq. (6) (and not only the Wiener processes), the proof above is true for any independent race model with any such processes as the accumulators.

Remark 3. Conditions similar to what we derived in Eq. (12) were first derived by Townsend (1976) in the context of the mimicry between serial and parallel models. He also proposed two sufficient conditions on $g_1(t)$ and $g_2(t)$ that assure $\lim_{t\to\infty} S_i(t) = 0$ (Theorem 2, parts B and C in that paper).

6. Computing time-varying boundaries

Now that we have proved that for a given race model we can always determine its boundaries such that it mimics a given Wiener process model, the next question is how to compute these boundaries. Given the results in the previous section, we can summarize this problem as follows:

Consider a given Wiener process model with the p.d.f.s of the FPT $g_1(t)$ and $g_2(t)$, and an independent race model with parameters $(\mu_1, \mu_2, \sigma_1, \sigma_2)$. Compute two functions $b_1(t)$ and $b_2(t)$ such that the p.d.f. of the FPT of the process:

- *X_1 with parameters (μ_1, σ_1) through the boundary $b_1(t)$ is $f_1(t) = p \cdot v(t) \cdot [\overline{V}(t)]^{(p-1)}$*
- *X_2 with parameters (μ_2, σ_2) through the boundary $b_2(t)$ is $f_2(t) = (1-p) \cdot v(t) \cdot [\overline{V}(t)]^{(-p)}$.*

Therefore, this problem consists of two separate (1-dimensional) inverse FPT problems. Several numerical methods have been proposed for solving this type of inverse FPT problem (Abundo, 2006, 2013; Song & Zipkin, 2011; Zucca & Sacerdote, 2009). We employ an approach similar to that proposed by Zucca and Sacerdote (2009). Here, we explain the algorithm for

Algorithm 1 Compute the boundaries of an independent race model to mimic a Wiener process model

The parameters $\mu, \sigma, \mu_1, \sigma_1, \mu_2, \sigma_2$ are given
Compute $g_1(t)$ and $g_2(t)$ using eq. (A2a) and (A2b) in Ratcliff and Smith (2004)
$f_1(t) = p \cdot v(t) \cdot [\overline{V}(t)]^{(p-1)}$ and $f_2(t) = (1-p) \cdot v(t) \cdot [\overline{V}(t)]^{(-p)}$
for k=1 to N **do**
  solve equations 15 for $b_1(k\Delta_t)$
end for

computing $b_1(t)$. The basic idea is to solve, at each time step k, the so-called discretized *Volterra integral equation* for $b_1(k\Delta_t)$:

$$\begin{aligned} f_1(k\Delta_t) &= -2\Psi[b_1(k\Delta_t)|0,0] + 2\Delta_t \sum_{j=1}^{k-1} f_1(j\Delta_t) \\ &\quad \times \Psi[b_1(k\Delta_t)|b_1(j\Delta_t), j\Delta_t] \\ \Psi\left[b_1(k\Delta_t)|y, j\Delta_t, k, b_1((k-1)\Delta_t)\right] &= \frac{0.5}{\sqrt{2\pi\sigma^2(k-j)\Delta_t}} \\ &\quad \times \exp\left[-\frac{(b_1(k\Delta_t) - y - \mu_1(k-j)\Delta_t)^2}{2\sigma^2(k-j)\Delta_t}\right] \\ &\quad \times \left[\frac{b_1(k\Delta_t) - b_1((k-1)\Delta_t)}{\Delta_t}\right. \\ &\quad \left. - \mu_1 - \frac{b_1(k\Delta_t) - y - \mu_1(k-j)\Delta_t}{(k-j)\Delta_t}\right]. \end{aligned} \tag{15}$$

Note that in this set of nonlinear equations, the only unknown variable at each time step k is $b_1(k\Delta_t)$. To solve for b at each time step, we discretize a plausible range of values for b and find the value that minimizes the difference between the two sides of Eq. (15). Algorithm 1 shows the pseudo-code for computing the boundaries.

It is worth noting that the functions $f(.)$ and $\Psi(.)$ in Eq. (15) are exactly the same as those used in the direct FPT problem to compute the p.d.f. of the FPT of a Wiener process with parameters (μ, σ) through the time-varying boundary $b_1(t)$ (see Smith, 2000 for a comprehensive review of the methods for solving the direct FPT problem). The main difference is that in the direct FPT problem, the boundary $b_1(t)$ is known and the equations are solved for the p.d.f. $f_1(t)$ while in the inverse FPT problem, $f_1(t)$ is known and the equations are solved for the unknown boundary $b_1(t)$.

Fig. 2 shows the results of applying this algorithm to compute the boundaries. The parameters of the Wiener process model used to generate this figure are $\mu = 0.1$, $\sigma = 0.1$. The boundaries of the Wiener process model are set at ± 0.09. To generate this figure, we first computed the p.d.f.s of the FPTs of the Wiener process through the given boundaries ($g_1(t)$ and $g_2(t)$) using Eqs. (A2a) and (A2b) in Ratcliff and Smith (2004). The top row shows the results for an independent race model with parameters $\mu_1 = -\mu_2 = 0.1$, $\sigma_1 = \sigma_2 = 0.1$. The left panel shows the estimated boundaries and the right panel shows the p.d.f.s of the FPTs predicted by the Wiener process model and the independent race model. It can be seen that the independent race model mimics the Wiener process model. It is shown that the boundaries $b_1(t)$ and $b_2(t)$ are decreasing functions of time and also $b_1(0) = b_2(0) = 0.09$. To solve Eq. (15), based on part (b) of the theorem, we set $b_1(0) = b_2(0) = a$ and then solve for $b_i(k\Delta_t)$ for all $k = 1, 2, \ldots$. This shows the importance of this part of the theorem.

The bottom panel of this figure shows the results for an independent race model with parameters $\mu_1 = 0.1$, $\mu_2 = 0$, $\sigma_1 = \sigma_2 = 0.1$. It can be observed that the shape of the boundaries is different from the top panel. As the right panel indicates, this model also mimics the Wiener process model perfectly.

As it can be seen in the left panels of Fig. 2, the mimicking boundaries for the correct and incorrect responses are different (the red and the black curves). For example, consider the independent race model used in the top panel of the figure. In this model, if the stimulus 1 (2) is presented in a trial the drift rate in accumulator 1 (2) is 0.1 while the drift rate in accumulator 2 (1) will be 0. In computing the boundaries, we assume that $b_1(t)$ is the boundary for the accumulator with drift rate equal to 0.1 and $b_2(t)$ is the boundary for the accumulator with drift rate equal to 0. Since the drift rates in the accumulator corresponding to the correct and incorrect responses in each trial are 0.1 and 0, respectively, $b_1(t)$ corresponds to the correct responses while $b_2(t)$ corresponds to the incorrect responses.

Having different decision boundaries for correct and incorrect responses is not a desirable property of a model. This property implies that before the trial starts, the subject should know the correct response and set its corresponding decision threshold to $b_1(t)$ and the decision threshold of the incorrect response accumulator to $b_2(t)$. But of course the subject does not know the correct response a priori and so this property is not plausible. Therefore, it is desirable to come up with a mimicking model in which the decision boundaries are symmetric. The remaining part of this section is devoted to developing a race model that can mimic a Wiener process model and has symmetric boundaries for correct and incorrect responses.

In the mimicking independent race model, the drift coefficients are constant while the boundaries are time-varying. To develop the model with boundaries which are not affected by the value of the drift rates, we use a simple technique: we take the time-variance in the boundaries out and put it in the drift coefficients. Consider, for example, the accumulator 1 in the mimicking model in the top panel of Fig. 2. The drift coefficient in this accumulator is $\mu_1 = 0.1$ and its decision boundary is the function $b_1(t)$ shown in the left-top panel. First, we decompose $b_1(t)$ into a constant and a time-varying part, that is $b_1(t) = c_0 + \beta_1(t)$ where c_0 is a constant and $\beta_1(t) = b_1(t) - c_0$. Let T_1 denote the FPT of the process X_1 (accumulator 1) through the boundary $b_1(t)$ and let $F_1(t)$ denote the CDF of this random variable. We have:

$$\begin{aligned} F_1(t) &= \Pr\left[T_1 \le t\right] = \Pr\left[X_1(\tau) \ge b_1(\tau);\, \tau \in (0,t]\right] \\ &= \Pr\left[X_1(\tau) - \beta_1(\tau) \ge c_0;\, \tau \in (0,t]\right]. \end{aligned} \tag{16}$$

It can be shown that (Smith, 2000, equation 17):

$$X_1(t) = \int_0^t \mu_1 d\tau + \sigma B(t) \tag{17}$$

where $B(t)$ is the Brownian motion process. Substituting for X_1 from Eq. (17) into Eq. (16) yields:

$$\begin{aligned} F_1(t) &= \Pr\left[\int_0^\tau \mu_1 d\nu + \sigma B(\tau) - \beta_1(\tau) \ge c_0;\, \tau \in (0,t]\right] \\ &= \Pr\left[\int_0^\tau (\mu_1 - \beta_1'(\nu))d\nu - \beta_1(0)\right. \\ &\quad \left. + \sigma B(\tau) \ge c_0;\, \tau \in (0,t]\right] \end{aligned} \tag{18}$$

where $\beta_1'(t) = \frac{d}{dt}\beta_1(t)$ (we assume that this derivative exists). If we set $c_0 = b_1(0)$ then $\beta_1(0) = 0$ which yields $F_1(t) = \Pr\left[\int_0^\tau(\mu_1 - \beta_1'(\nu))d\nu + \sigma B(\tau) \ge c_0;\, \tau \in (0,t]\right]$. The process $Y_1(\tau) = \int_0^\tau(\mu_1 - \beta_1'(\nu))d\nu + \sigma B(\tau)$ is a time-inhomogeneous Wiener process with time-varying drift coefficient $\mu_1 - \beta_1'(t)$ and constant diffusion coefficient σ. Eq. (18) shows that $F_1(t)$ is the CDF of the FPT of this process through the constant boundary c_0.

In sum, these results show that the CDF (and so the p.d.f.) of the FPT of a Wiener process with parameters (μ_1, σ) through the boundary $c_0 + \beta_1(t)$ is equal to that of a Wiener process with the

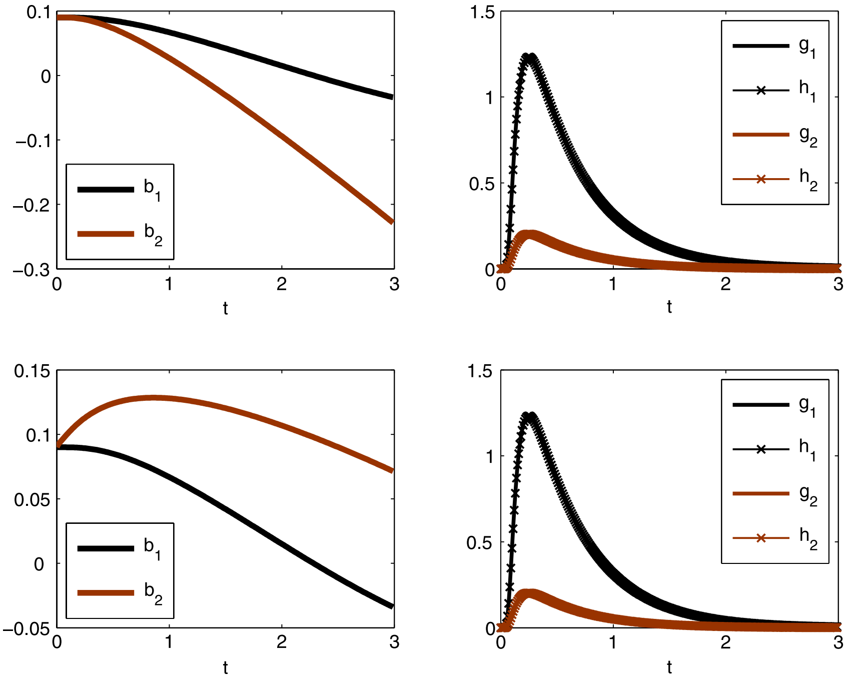


Fig. 2. The results of applying Algorithm 1 to compute the boundaries of the independent race model. Left: the computed boundaries. Right: the p.d.f. of the FPTs of the two responses predicted by the Wiener process model ($g_1(t)$ and $g_2(t)$) and the independent race model ($h_1(t)$ and $h_2(t)$). The parameters of the Wiener process model used are: $\mu = 0.1, \sigma = 0.1, a = 0.09$. The parameters of the independent race model in the top row are $\mu_1 = -\mu_2 = 0.1, \sigma_1 = \sigma_2 = 0.1$ and in the bottom row the parameters are $\mu_1 = 0.1, \mu_2 = 0, \sigma_1 = \sigma_2 = 0.1$.

drift coefficient $\mu_1 - \beta_1'(t)$ and the diffusion coefficient σ through the constant boundary c_0 (see Smith, 1995, 2000 for a thorough discussion). Similar results can be derived for the accumulator 2. For the independent race model used in the top panel of Fig. 2, the drift coefficient of accumulator 2 is 0. We can show that the CDF of this Wiener process through the boundary $b_2(t) = c_0 + \beta_2(t)$ is equal to that of a Wiener process with the drift coefficient $-\beta_2'(t)$ and diffusion coefficient σ through the constant boundary c_0.[6]

This shows that an independent race model with parameters (μ_1, μ_2, σ) and decision boundaries $b_1(t)$ and $b_2(t)$ is equivalent to an independent race model with drift coefficients $\mu_1 - \beta_1'(t)$ and $\mu_2 - \beta_2'(t)$, diffusion coefficients σ, and the decision boundaries c_0. This model is depicted in the middle panel of Fig. 3.

Although the decision boundaries are symmetric in this model, it assumes that the drift rates in the two accumulators are two different time-varying functions. We must specify the relationship between these functions and the physical properties of the stimulus. In the Wiener process model, it is assumed that the drift coefficient is proportional to the discriminability of the stimulus. In the independent race model used in the bottom panel of Fig. 2, the drift coefficient of the accumulator corresponding to the presented stimulus is proportional to the discriminability while the drift coefficient of the other accumulator is zero. In the top panel of Fig. 3, this is shown by sending the values μ_1 and 0 to the two accumulators, respectively. To specify the relationship between the discriminability of the stimulus and the time-varying drift coefficients in the mimicking race model with constant boundaries (the middle panel of Fig. 3), we propose the architecture shown in the bottom panel of Fig. 3. The part of the model which is inside the gray box is exactly the same as the model in the middle panel of the figure. However, the values μ_1 and 0 are multiplied by two dynamic gains $K_1(t)$ and $K_2(t)$ and then summed before the noise is added. In this model we have:

$$\begin{aligned} &\mu_1 \cdot K_1(t) + 0 \cdot K_2(t) = \mu_1 - \beta_1'(t) \\ &0 \cdot K_1(t) + \mu_1 \cdot K_2(t) = -\beta_2'(t) \end{aligned} \tag{19}$$

solving this equations yields:

$$\begin{aligned} K_1(t) &= 1 - \frac{\beta_1'(t)}{\mu_1} \\ K_2(t) &= -\frac{\beta_2'(t)}{\mu_1}. \end{aligned} \tag{20}$$

Although $K_1(t) \neq K_2(t)$, this model is symmetric because both accumulators have the same gains and the same decision thresholds. Another important point is that the two accumulators interact in this model. However, all the interaction happens before the noise is added and so the stochastic part of the accumulated information remains independent.

Based on the results obtained in this section, we can state the theorem proved in Section 5 for the model shown in the bottom panel of Fig. 3: consider a Wiener process model with parameters (μ, σ) and the decision boundaries $\pm a$. For all values of μ, a and all $\sigma \geq 0$, if we set $c_0 = a$ and $\mu_1 = \mu$, there *exist* two *unique* functions $K_1(t)$ and $K_2(t)$ such that the model shown in the bottom panel of Fig. 3 mimics the Wiener process model.

In most of the perceptual decision making experiments, stimuli with several levels of discriminability are intermixed and in each trial one discriminability level is randomly picked and presented. When fitting sequential sampling models to data from these experiments, it is usually assumed that the subject adopts one set of decision thresholds for all discriminability levels. However, the drift coefficient is different for different discriminability levels (Palmer, Huk, & Shadlen, 2005; Ratcliff, 1978; Ratcliff & Smith, 2004; Ratcliff et al., 1999). Therefore, it is interesting to investigate in the mimicking race model how the gains $K_1(t)$ and $K_2(t)$ change when in the Wiener process model the decision boundary a is kept fixed and the drift coefficient μ changes. The results of such

6 Note that since $b_1(0) = b_2(0) = a$, if we set $c_0 = a$ then $\beta_1(0) = \beta_2(0) = 0$.

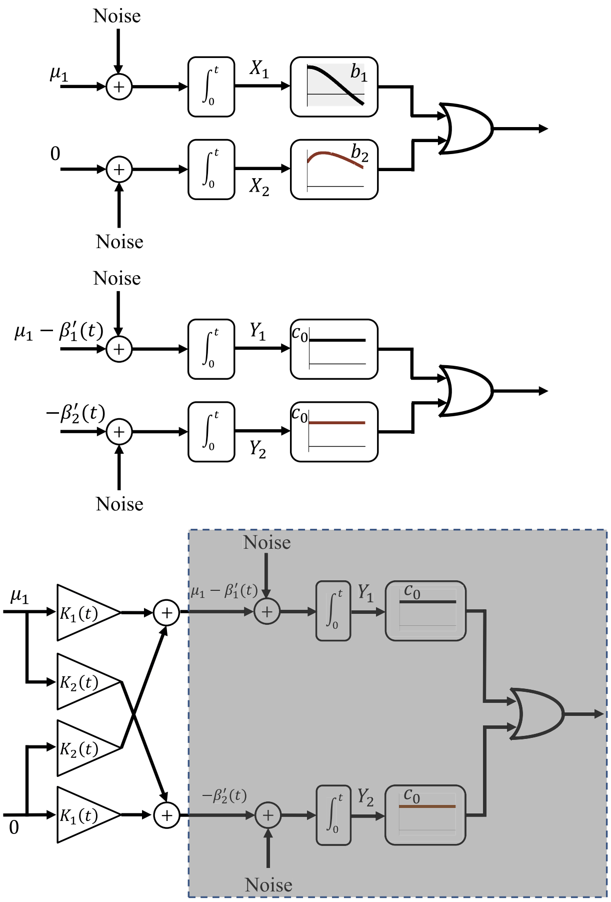


Fig. 3. Three equivalent race models. Top: an independent race model with constant drift coefficients and time-varying boundaries. Middle: an independent race model with constant symmetric boundaries and time-varying drift coefficients. Bottom: an interactive race model with constant symmetric boundaries. See the text for more details.

analysis is shown in Fig. 4. In this figure, we have considered a Wiener process model with $\sigma = 0.1$, $a = 0.09$ and several values of μ shown in the figure. Then, for each value of μ we have computed the functions $K_1(t)$ and $K_2(t)$ such that the model shown in the bottom panel of Fig. 3 with $\mu_1 = \mu$, $\sigma_1 = \sigma_2 = 0.1$ and $c_0 = a = 0.09$ mimics the Wiener process model. As seen, the gains depend not only on time but also on the value of μ.

7. Relation to general independent race models

In the independent race model considered in this paper, each accumulator is modeled as a Wiener process and the decision boundaries are deterministic. This model falls into a more general class of independent race models which, for the sake of clarification, we call general independent race models. Previous research has investigated the properties of the p.d.f.s of reaction times that can be produced by these models (Jones & Dzhafarov, 2014; Marley & Colonius, 1992; Townsend, 1976). For completeness, we first define this class of models and then state the most pertinent results on their properties to the results presented in this paper and finally explain how they are related. Here, we only present the definition and results for the case in which there are only two possible responses, one condition and one stimulus value in the experiment.

Definition (*General Independent Race Model*)**.** In a general race model, each response i, $i = 1, 2$, is associated with an *information processing channel* which is specified by a stochastic processes $R_i(t)$ and a threshold, θ_i, which is a random variable. The FPT in each channel without considering other channels is $T_i = \inf\{t > 0 | R_i(t) \geq \theta_i\}$. If the random variables T_i are independent, then the model is a general independent race model.

The reaction time in each trial is $\widehat{T} = \min\{T_i\}$. We denote the p.d.f. of giving the response i at time t by $h_i(t)$ and its CDF by $H_i(t) = \Pr[\widehat{T} \leq t, i]$. The following theorem specifies the family of reaction time distributions that can be produced by this class of models.

Theorem (Jones & Dzhafarov, 2014)**.** *Let $g_1(t)$ and $g_2(t)$ be two p.d.f.s of reaction time, $G(t) = \int_0^t (g_1(\tau) + g_2(\tau))d\tau$ be the CDF of the marginal reaction time and $\lambda_i(t) = \frac{g_i(t)}{1-G(t)}$ be the hazard function of the ith response. Let $M = \big((R_1(t), \theta_1), (R_2(t), \theta_2)\big)$ be a general independent race model. If $\lambda_i(t) = 0$ for $t > t_{max}$ then M is a general independent race representation of $g_1(t)$ and $g_1(t)$ (that is $h_i(t) = g_i(t),\ i = 1, 2$) if and only if:*

$$\Pr[T_i \leq t] = 1 - \exp\left(-\int_0^t \frac{g_i(\tau)}{1 - G(\tau)d\tau}\right) \tag{21}$$

for all $t < t_{max}$. Eq. (21) provides the necessary and sufficient condition for the set of distributions that can be produced by the general independent race models.[7]

There is an important difference between the results we obtained in Section 5 and this theorem: in Jones and Dzhafarov's theorem, there are no constraints on the processes $R_i(t)$. For any given pair of $g_1(t)$ and $g_2(t)$, the processes $R_i(t)$ and the thresholds θ_i should be chosen such that Eq. (21) is satisfied. In contrast, in the theorem of Section 5, we restricted the processes $R_i(t)$ to be Wiener.

A more restricted version of the general independent race model is the Grice framework with independent thresholds in which $R_i(t)$ are not stochastic processes but deterministic functions (Dzhafarov, 1993; Jones & Dzhafarov, 2014) and the thresholds θ_i are independent random variables. Dzhafarov (1993) showed that for any joint distribution on the decision thresholds, there exist deterministic functions $R_i(t)$ such that $M = \big((R_1(t), \theta_1), (R_1(t), \theta_1)\big)$ is a Grice representation of two given distributions $g_1(t)$ and $g_2(t)$. Another restricted version of the general independent race model is the *ballistic accumulator* model (Brown & Heathcote, 2005; Jones & Dzhafarov, 2014). In this model, $R_i(t) = k_i L_i(t)$ (or $R_i(t) = k_i + L_i(t)$) where k_i's are random variables and $L_i(t)$'s are deterministic functions. Also θ_i's, are deterministic and all have the same value, say b. Jones and Dzhafarov (2014) proved that under some technical assumptions, for any given $g_1(t)$ and $g_2(t)$ and functions $L_i(t)$, there exist random variables k_i such that $M = \big((k_1 L_1(t), b), (k_2 L_2(t), b)\big)$ is an independent race representation of $g_1(t)$ and $g_2(t)$. The main difference between our results and these theorems is that in all these models R_i's are either deterministic or random variables and not stochastic processes. In contrast, the accumulators are Wiener processes in the independent race model we considered. Due to this distinction, in order to prove our theory we had to show that there exists a unique solution to the corresponding inverse FPT problem. In general, if we assume that $R_i(t)$'s are specific stochastic processes, to show that Eq. (21) is satisfied for given distributions $g_1(t)$ and $g_2(t)$, one should solve the corresponding inverse FPT problem. This is what we did in the

[7] It is important to note that this equation is equivalent to Eq. (12) and the constraint $S_1(\infty)S_2(\infty) = 0$ should also be satisfied here. As we mentioned in Section 5, this type of conditions were first developed by Townsend (1976). In our proof, we followed the steps in Townsend (1976) because this proof makes the necessity of the constraint clearer. It also shows the connection between our theorem and the problem of mimicry between serial and parallel models.

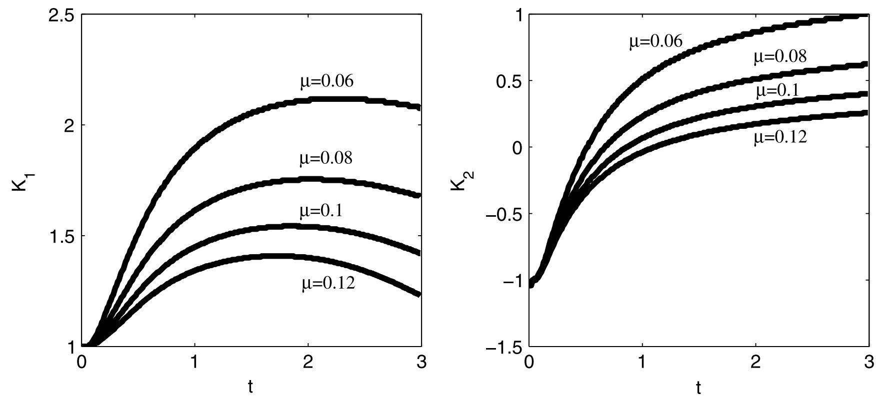


Fig. 4. Time-varying gains $K_1(t)$ (left) and $K_2(t)$ (right) for different values of μ. To compute these gains, for each value of μ we set $\mu_1 = \mu$ in the model shown in the bottom panel of Fig. 3. We also set $c_0 = a$ and $\sigma_1 = \sigma_2 = \sigma$ and then compute the gains such that model mimics a Wiener process model with parameters (μ, σ, a). The parameters used are: $\mu = 0.1, \sigma = 0.1, a = 0.09$.

proof of the theorem in Section 5. As we explained before, however, the inverse FPT problem has not been solved except for a few cases (Abundo, 2013; Cheng et al., 2006).

Another point is that in the general independent race model and its more restricted variants, the thresholds are assumed to be constant. We showed, however, that if we restrict the processes $R_i(t)$ to be Wiener, the resulting boundaries will be time-varying (see the model in the top panel of Fig. 3).

8. Mimicking the full diffusion model

It can be shown that for all values of μ, σ and a the mean reaction times for the two responses in the Wiener process are the same. However, numerous experimental data show otherwise (Ratcliff, 1978; Ratcliff & Smith, 2004; Ratcliff et al., 1999; Usher & McClelland, 2001). Particularly, in some experimental conditions the mean reaction time for the correct responses is faster than the incorrect responses while in other conditions the opposite pattern has been observed. Ratcliff and his colleagues (Ratcliff, 1978; Ratcliff & Rouder, 1998; Ratcliff et al., 1999) extended the Wiener processes to a model which is able to predict these patterns. In this model, the drift coefficient and the starting value of the accumulated information are considered to be Gaussian and uniform random variables, respectively. The variability in the drift coefficient results in slower incorrect mean reaction time while variability in the starting value results in faster incorrect mean reaction time. Extensive research has shown that this model fits very well to the data from reaction time tasks. To distinguish this model from the Wiener process model we call it the *full diffusion model*. This model has some other parameters but we do not consider them here.

Our aim here is to use the Algorithm 1 to compute the boundaries of an independent race model to mimic a full diffusion model. We confine our analysis to the case where the drift coefficient is distributed as a Gaussian $N(\mu, \sigma_\mu^2)$ and there is no variability in the starting value.[8]

Algorithm 1 can be used to compute the mimicking boundaries. Fig. 5 shows the results of mimicry of a full diffusion model by an independent race model. The top panel of the figure shows the mimicking boundaries and the p.d.f. of the responses. The bottom panel shows the gains $K_i(t)$ in the model in the bottom panel of Fig. 3 for different values of σ_μ. As it can be seen, the gains change dramatically when this parameter varies.

9. Mimicking Wiener process model by Ornstein–Uhlenbeck process model

In this section, we consider the problem of mimicking a given Wiener process model by an Ornstein–Uhlenbeck (OU) process model. The OU process model has been used extensively in modeling different decision making problems (Busemeyer & Diederich, 2002; Busemeyer & Townsend, 1993). This process is a continuous Markov process and so is a special case of Eq. (6). Specifically, this process is specified by setting $\Psi(X(t), t) = \mu - \gamma \cdot X$ and $\Phi(X, t) = \sigma$ where $\gamma > 0$ is called the decay rate.

Unfortunately, unlike the case of the independent race model, we have not been able to prove the existence and uniqueness of the decision boundaries for an OU process with given drift, diffusion, and decay parameters to mimic a given Wiener process model. However, our extensive simulations with many different parameter values provide strong evidence for such mimicry of the Wiener process model by the OU process model. We state these results in the form of a conjecture.

Conjecture. *Consider a Wiener process X with parameters (μ, σ) and two constant boundaries $\pm a$. Also, consider an OU process X_γ with parameters (μ, σ, γ). Then, for all given values of μ, $\sigma > 0$ and $\gamma > 0$, there exist two unique functions $b_1(t)$ and $b_2(t)$ such that the p.d.f.s of the FPT of the process X_γ through the boundaries $b_1(t)$ and $b_2(t)$ are equal to the p.d.f.s of the FPT of the process X through the boundaries $\pm a$.*

Algorithm 1 can be easily extended to the case of computing the mimicking boundaries of an OU processes. Fig. 6 shows the results of such algorithm to compute the boundaries of the OU process. It is hard to see in the figure, but the resulting boundaries are slightly asymmetric. Intuitively this makes sense because, as we explained, the mean reaction times for the correct and incorrect responses are equal in a Wiener process model while this is not the case in an OU process. Hence, to be able to mimic the Wiener process model the boundaries of the OU process should be asymmetric. In contrast to the race model, it is not possible to come up with a symmetric mimicking OU model. Also, it is important to note that the shape and asymmetry of the boundaries depend on the value of the drift coefficient μ. Therefore, similar to the situation we showed in Fig. 4, to mimic a Wiener process model of an experiment with several levels of the stimulus discriminability, we should compute different mimicking boundaries for each value of μ.

As seen in the figure, even for large values of γ, the resulting OU process model mimics the Wiener process model with the given parameters. The resulting boundaries are decreasing and $b_1(0) = -b_2(0) = 0.9$. For larger values of γ the boundaries converge to values closer to zero.

[8] It is important to note that since when $\sigma_\mu > 0$ then $\frac{g_1(t)}{g_2(t)} \neq \frac{p}{1-p}$, the proof presented in Section 5 is not valid anymore.

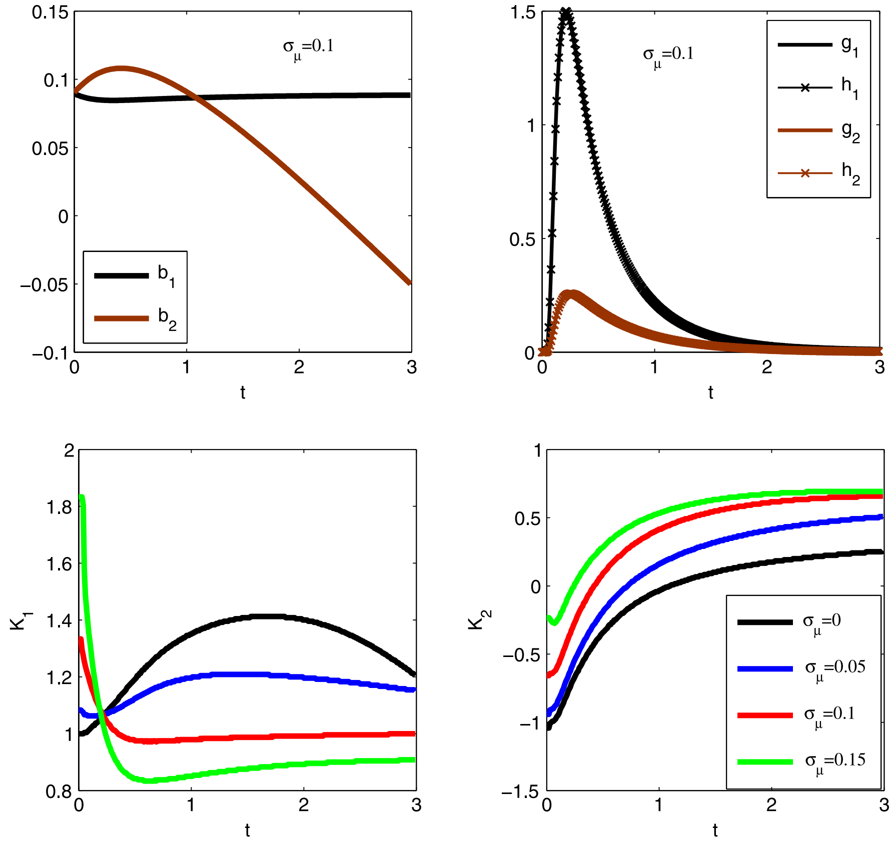


Fig. 5. Mimicking a full diffusion model with an independent race model. Top left: the mimicking boundaries. Top right: the p.d.f. of the reaction times predicted by the full diffusion model and the independent race model. Bottom: The corresponding gains $K_1(t)$ (left) and $K_2(t)$ (right) in the model in the bottom panel of Fig. 3 for different values of σ_μ. Other parameters are: $\mu = 0.1$, $\sigma = \sigma_1 = \sigma_2 = 0.1$, $a = 0.09$.

10. Discussion

The main contribution of this paper was to investigate the mimicry between several sequential sampling models, both theoretically and numerically. By casting the problem of mimicry as an inverse FPT problem, we were able to prove that a given Wiener process model with symmetrical boundaries can always be mimicked by a given independent race model with appropriate decision boundaries. We showed how these boundaries can be computed. The numerical computation of the boundaries showed that the mimicking boundaries are time-varying and asymmetric. We then proposed a symmetric mimicking race model in which the decision boundaries are constant but the stimulus discriminability is multiplied by time-varying gains.

Race models with time-varying gains have been proposed previously (Cisek, Puskas, & El-Murr, 2009; Ditterich, 2006). In these models, the gains are parametric functions of time and the parameters are estimated by fitting the models to the data. In the mimicking race model we proposed, the gains are obtained by solving the corresponding Volterra integral equation and do not take a parametric form. However, it would be interesting to see if we can approximate the gains with parametric functions. In that case, the resulting model will not mimic the Wiener process model exactly and we should assess the discrepancy between the two models for different values of the parameters. Another important difference between the model in the lower panel of Fig. 3 and the models in Cisek et al. (2009) and Ditterich (2006) is that in our model the gains are multiplied before the noise is added to the signals. In other words, the interaction in our model happens in the physical stimulus and the stochastic parts of the model remain independent.

Recently, Jones and Dzhafarov (2014) showed that the race model is universal in the sense that if we do not restrict the accumulators to have a specific form, the race model can generate any form of the distributions of reaction time. The important difference between our results with their results is that in the independent race model we considered in this paper the accumulators are restricted to the Wiener processes. This restriction is important, because this version of the race model is consistent with the neuro-physiological findings which show that the activation of some neuron populations during perceptual decision making resembles a form of stochastic accumulation (Gold & Shadlen, 2002, 2007; Roitman & Shadlen, 2002).

The theorem we proved and Remark 2 in Section 5 show that the independent race model with Wiener process accumulators is still universal. Jones and Dzhafarov (2014) used the universality results to argue that the race models with no restrictions are not falsifiable. These results show that by relaxing the assumption of constant decision boundaries, models like the independent race model and the OU model can be good competitors to the Wiener process models. This motivates the development of more sophisticated experimental designs that can distinguish between these models (Khodadadi, Fakhari, & Busemeyer, 2014; Teodorescu & Usher, 2013; Tsetsos et al., 2012).

As we mentioned in the Introduction, another approach for comparing sequential sampling models is to fit these models to empirical data and compare them using some statistical measures of goodness of fit. For example, in their seminal work, Ratcliff and Smith (2004) fitted several sequential sampling models to the same data sets and compared the goodness of fit of each model to each data set (see also Smith & Ratcliff, 2009 for a similar comparison between a version of the Wiener process model with time-varying drift and diffusion coefficients and a

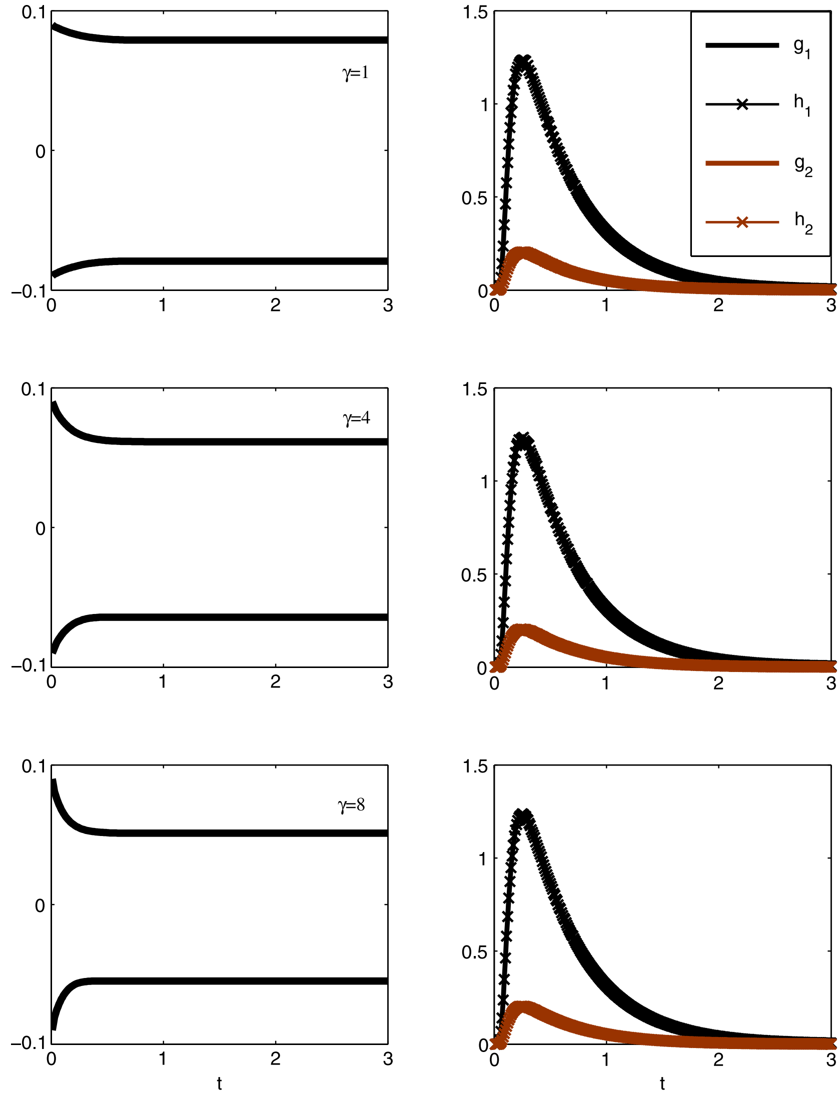


Fig. 6. The results of applying an algorithm analogous to Algorithm 1 to compute the boundaries of an OU process with different values of γ. Left: the computed boundaries. Right: the p.d.f. of the FPTs of the two responses predicted by the Wiener process model ($g_1(t)$ and $g_2(t)$) and the OU process model ($h_1(t)$ and $h_2(t)$). The parameters used are: $\mu = 0.1$, $\sigma = 0.1$, $a = 0.09$.

version of the independent race model with the OU process accumulators). One of their findings pertinent to our results is that for small to moderate values of the decay rate γ, the predictions of the OU process model and the Wiener process model (both with constant boundaries) are indistinguishable. In other words, although the two models do not mimic each other exactly, they are empirically indistinguishable. Their results showed that for large values of the decay rate parameter, however, the two models are distinguishable and the predictions of the OU process model are not consistent with the data. Our numerical example in Section 9 (Fig. 6) shows that for larger values of γ, the difference between the initial and the asymptotic values of the mimicking boundaries is larger and so their approximation with time-constant decision boundaries will not be accurate.

As it can be seen in Fig. 4, the gains $K_1(t)$ and $K_2(t)$ have rather complicated forms. The self-excitation term $K_1(t)$ increases first then decreases while the cross-excitation term $K_2(t)$ increases. Similar complicated patterns have been observed in Jones and Dzhafarov (2014) (see for example Fig. 5 in that paper) and Zhang et al. (2014) (see for example Fig. 3 in that paper). This raises the question of whether these mimicking models are psychologically and/or neurophysiologically plausible. We are not aware of any behavioral or neural evidence supporting the plausibility of such complicated patterns. However, it should be noted that these patterns arise when we try to build models that perfectly mimic a Wiener process model. But there is no reason to believe that the Wiener process model is the best model for perceptual decision making tasks. We may find simpler and more plausible forms for the gains $K_1(t)$ and $K_2(t)$ such that the resulting model approximately mimics a Wiener process model and fits reasonably to the data. For example the gains could be considered as the output of a set of linear filters (see for example Smith, 1995). Our results show that a race model can become arbitrarily similar to a Wiener process model. Further research is necessary to see if there is a plausible approximation of the mimicking race model that can fit the data well.

Appendix

In this appendix, we prove that for any values of the parameters $(\mu, \sigma, a, \mu_1, \mu_2)$ in the theorem proved in Section 5 if $\sigma_1 = \sigma_2 = \sigma$ then $\lim_{t\to 0} b_i(t) = a$. We show that the proof for $i = 1$ and the proof for $i = 2$ will be similar.

First we note that as Δ_t goes to zero the first order approximation of $\overline{V}$ is $\overline{V}(\Delta_t) \approx 1 - \Delta_t \cdot v(\Delta_t)$. Since $f_1(t) = p \cdot v(t) \cdot$

$[\overline{V}(t)]^{(p-1)}$, as Δ_t goes to zero we have $f_1(\Delta_t) \approx p \cdot v(\Delta_t) = g_1(\Delta_t)$. On the other hand we have:

$$f_1(\Delta_t) = -2\Psi[b_1(\Delta_t)|0,0] = \frac{-1}{\sqrt{2\pi\sigma^2\Delta_t}} \exp\left(-\frac{(b_1(\Delta_t)-\mu_1\Delta_t)^2}{2\sigma^2\Delta_t}\right) \times \left[{b_1}'(\Delta_t) - \mu_1 - \frac{b_1(\Delta_t)-\mu_1\Delta_t}{\Delta_t}\right] \tag{A.1}$$

and:

$$g_1(\Delta_t) = -2\Psi[a|0,0] = \frac{-1}{\sqrt{2\pi\sigma^2\Delta_t}} \exp\left(-\frac{(a-\mu\Delta_t)^2}{2\sigma^2\Delta_t}\right) \times \left[-\mu - \frac{a-\mu\Delta_t}{\Delta_t}\right]. \tag{A.2}$$

Eq. (A.1) is obtained from the Volterra integral equation of a Wiener process with boundary $b_1(t)$ (Eqs. 27, 47 and 57 in Smith, 2000). Similarly, Eq. (A.2) is obtained from the Volterra integral equation of a Wiener process with boundaries $\pm a$. Setting $f_1(\Delta_t) = g_1(\Delta_t)$ and ignoring the terms $\mu \cdot \Delta_t$ in Eqs. (A.1) and (A.2) (because $\Delta_t \to 0$) yields:

$$\exp\left(-\frac{b_1^2(\Delta_t)}{2\sigma^2\Delta_t}\right) \times \left[{b_1}'(\Delta_t) - \frac{b_1(\Delta_t)}{\Delta_t}\right] = \exp\left(-\frac{a^2}{2\sigma^2\Delta_t}\right) \times \left[-\frac{a}{\Delta_t}\right]. \tag{A.3}$$

Taking the logarithm from both sides of this equation yields:

$$b_1^2(\Delta_t) = a^2 + \log\left[\frac{b_1(\Delta_t)}{a} - \frac{{b_1}'(\Delta_t)}{a}\Delta_t\right] \cdot 2\sigma^2\Delta_t. \tag{A.4}$$

The second term in the right hand side of this equation goes zero when $\Delta_t \to 0$ and so $\lim_{\Delta_t \to 0} b_1^2(\Delta_t) = a^2$. Since the argument of the log function should be positive we should have $\frac{b_1}{a} > 0$ and so $\lim_{\Delta_t \to 0} b_1(\Delta_t) = a$. This completes the proof. □

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